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EXPERIMENTS ON EFFECTIVE ELASTIC MODULUS OF TWO-DIMENSIONAL SOLIDS WITH CRACKS AND HOLES

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Abstract—In this paper, some basic concepts on effective properties of nonhomogeneous elastic solids are reviewed. The various theories are evaluated by conducting experiments on artificially cracked and porous solids, and by comparing the results with the theoretical predictions for the cases of interacting and non-interacting inhomogeneities. Two aluminum plates containing slots and two containing circular holes in random mutual positions with different orientational distributions were tested. In the case of plates containing random slots it is shown that the approximation of non-interacting cracks provides a reasonable estimate of the modulus, even when interactions may be significant. The results obtained from the plates containing circular holes indicate that, as porosity increases, the effective Young's modulus follows the predictions for the case of interacting holes from both Mori-Tanaka and differential schemes. Copyright © 1996 Elsevier Science Ltd.

NOTATION

a	semi-major axis of ellipse
A	reference area
α_{ij}	crack density tensor
b	semi-minor axis of ellipse
β_{ij}	holes' density tensor
δ_{ij}	Kronecker delta
$\Delta\varepsilon_{ij}$	additional strain tensor due to the presence of a hole
$\Delta\varepsilon_{v}$	additional volumetric strain due to the presence of a hole
Δf	change in elastic potential due to holes
$(e_1)_i$, $(e_2)_i$	eigenvectors of β
E	effective Young's modulus
E_0	Young's modulus of matrix material
ε_{ij}	average strain tensor
f	elastic potential of material containing holes
f_0	elastic potential of matrix material
Γ	hole boundary
H_{ijkl}	hole's compliance tensor
l	half crack length
m_i	unit normal to minor axis of ellipse
M^0_{ijkl}	compliance tensor of matrix material
M^{eff}_{ijkl}	effective compliance tensor
n_i	unit normal to major axis of ellipse
v	effective Poisson's ratio
v'_0	Poisson's ratio of matrix material
p	porosity
P	isotropic stress
q	eccentricity parameter
ρ	crack density parameter
ρ_1, ρ_2	eigenvalues of β
σ_{ij}	average stress tensor
u_i	displacement of hole boundary

1. INTRODUCTION

The problem of determining effective elastic properties of inhomogeneous materials has applications in different fields such as materials science, structural mechanics and geophysics. The presence of inhomogeneities generally causes a reduction in the elastic moduli of a solid material. Therefore, a determination of effective properties, which take into account the effect of microcracked or porous structures, becomes important.

The objective of this work is to present some basic concepts on effective properties of elastic solids with cavities and cracks by reviewing the pioneering work of Kachanov (1993), as well as to evaluate the various approximate schemes by conducting well-designed experiments on artificially cracked and porous solids. The first part discusses some of the theories proposed for determining effective elastic properties of solids containing elliptical holes. The solutions are presented for cases of both interacting and non-interacting holes.

The second part of the paper presents the results of experiments designed to measure effective elastic properties of artificially cracked and porous aluminum plates under plane stress conditions. The plates contained slots or circular holes located in random mutual positions, for different orientational distributions. For each case, the effective Young's modulus was determined as a function of crack density (in the case of slots) or porosity (in the case of holes). The results obtained were compared with the predictions from the theories discussed for the cases of interacting and non-interacting inhomogeneities.

2. EFFECTIVE PROPERTIES OF SOLIDS WITH ELLIPTICAL HOLES

The analysis presented here follows that of Kachanov (1993) and has the advantage of being a general treatment, since it can describe the behavior of a material with an arbitrary distribution of holes of various aspect ratios. Cracks and circular holes are covered as limiting special cases. The key point is the identification of the proper parameters to describe the density of holes in the material. These parameters emerge naturally from the structure of the elastic potential: one scalar, the porosity p, and a second rank tensor β, which is called the holes' density tensor. The porosity vanishes in the case of cracks while the holes' density tensor reduces to the crack density tensor (α). In the case of circular holes, the tensor β is proportional to the unit tensor, and porosity then becomes the only density parameter. For any other aspect ratio of holes, the two parameters combined are needed for the complete description of the problem. Even when the orientational distribution of holes is random and the effective properties are isotropic, the effective moduli cannot be described only by porosity, since an evaluation of "eccentricity" is also needed. The problem is analyzed in the framework of linear elasticity. Non-linear effects due to holes closing are not considered; in the case of compressive stresses, this imposes some limitation on the magnitude of stresses (Zimmerman, 1991).

2.1. *Elliptical hole in a uniform stress field*

Consider the case where an ellipse of major axis $2a$ and minor axis $2b$ within a reference area A is subjected to a uniform stress $\sigma_{11} = \sigma_{22} \equiv P$. The additional volumetric strain $\Delta\varepsilon_v$ due to the presence of the hole can be calculated from a known solution of elasticity (Muskhelishvili, 1963), which yields

$$\Delta\varepsilon_v = \frac{1-2\nu_0}{1-\nu_0}\frac{P}{E_0}\frac{1}{A}2\pi(a^2+b^2) = \frac{1-2\nu_0}{1-\nu_0}\frac{P}{E_0}\frac{1}{A}2\pi[2ab+(a-b)^2], \tag{1}$$

where E_0 and ν_0 are the Young's modulus and Poisson's ratio of the matrix under plane stress conditions. This expression shows that, among elliptical holes with the same area (πab), the more elongated ones have the higher compressibility.

Figure 1 illustrates the compressibility of an elliptical hole as a function of the aspect ratio b/a for a fixed value of a. Near the limiting case of a crack ($b/a = 0$) the slope of the curve is approximately horizontal, which shows that a small "inflation" of a crack does not considerably affect the compressibility; at $b/a = 0.1$, the difference is only 1%. Near the limiting case of a circular hole a similar observation can be made, since the slope of the curve nearly coincides with the slope for a circular hole with the same area. For $b/a = 0.8$, the difference between the two solutions is less than 2.5%, showing that elliptical holes of high aspect ratios can be approximated by circular holes with good accuracy.

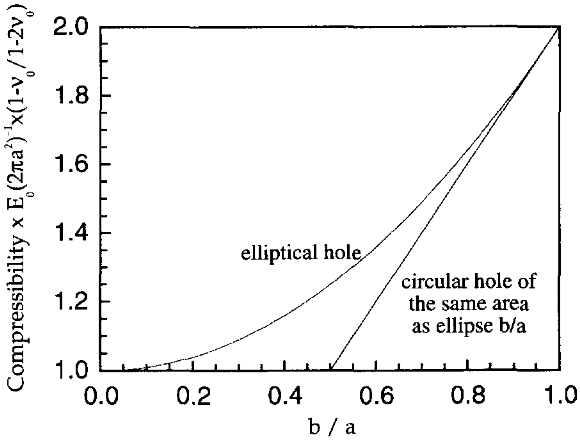

Fig. 1. Compressibility of an elliptical hole as a function of the aspect ratio (Kachanov, 1993).

2.2. *Non-interacting elliptical holes*

Now consider the case of a solid containing many elliptical holes and subjected to an externally applied stress σ. In the approximation of non-interacting holes, each hole is analyzed as an isolated one, subjected to the applied stress σ. The additional strain due to the presence of the holes is then a sum of the additional strains due to each individual hole, such that the effective compliance tensor can be written as

$$M^{eff}_{ijkl} = M^0_{ijkl} + \sum_k H^{(k)}_{ijkl}, \tag{2}$$

where M^0_{ijkl} is the compliance tensor of the matrix material and $H^{(k)}_{ijkl}$ is the hole compliance tensor for the k-th hole (Kachanov, 1993).

The elastic potential can be represented by the sum of two terms:

$$f(\sigma_{ij}) = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\sigma_{ij}M^0_{ijkl}\sigma_{kl} + \tfrac{1}{2}\sigma_{ij}H_{ijkl}\sigma_{kl} \equiv f_0 + \Delta f, \tag{3}$$

where f_0 represents the elastic potential of the intact material subjected to the loading σ and Δf represents the change in the elastic potential due to the presence of a hole. An explicit expression for Δf in terms of σ_{ij} can be obtained by substituting H_{ijkl} (Kachanov, 1993). The expression for Δf is then written as

$$\Delta f = \sigma_{ij}\sum_k H^{(k)}_{ijkl}\sigma_{kl} = \frac{1}{2E_0}\{p[4\sigma_{ij}\sigma_{ji} - (\sigma_{kk})^2] + 2\sigma_{ij}\sigma_{jk}(\beta_{ik} - p\delta_{ik})\}, \tag{4}$$

where it is expressed in terms of two dimensionless parameters related to the density of holes, one scalar (p) and one tensorial (β_{ik}). These parameters are defined as

$$p = \frac{1}{A}\pi\sum_k (ab)^{(k)}$$

$$\beta_{ik} = \frac{1}{A}\pi\sum_k (a^2 n_i n_k + b^2 m_i m_k)^{(k)}. \tag{5}$$

The scalar parameter p is the porosity of the material, which is the area of the elliptical holes divided by the total area. It does not depend on the orientational distribution of holes, and vanishes in the case of cracks. The tensor β is called the holes' density tensor and includes information on the orientational distribution of holes and their eccentricity. In the case of cracks of length $2a^{(k)}$, β_{ik}/π reduces to the two-dimensional crack density tensor $\alpha_{ik} = (1/A)\Sigma(a^2 n_i n_k)^{(k)}$, introduced by Vakulenko and Kachanov (1971) and

Kachanov (1980). The elastic potential of a solid with non-interacting cracks becomes $\Delta f = (\pi/E_0)\sigma_{ij}\sigma_{jk}\alpha_{ik}$.

Due to symmetry of the holes' density tensor, it can be written in two dimensions as

$$\beta_{ik} = \rho_1 (e_1)_i (e_1)_k + \rho_2 (e_2)_i (e_2)_k, \tag{6}$$

where ρ_1, ρ_2, e_1 and e_2 are the principal values and unit eigenvectors of β, respectively. This implies orthotropy of the effective properties, the axes of orthotropy coinciding with the principal axes of β.

The eccentricity of the holes is characterized by the difference $\beta_{ik} - p\delta_{ik}$, which can be called the eccentricity tensor. Its linear invariant, $tr(\beta_{ik} - p\delta_{ik})$, is given by

$$q = \beta_{ii} - 2p = \pi \frac{1}{A} \Sigma (a-b)^2, \tag{7}$$

where q is called the eccentricity parameter. It vanishes in the case of circular holes ($a = b$). In the case of cracks, q/π reduces to the conventional scalar crack density parameter $\rho = (1/A)\Sigma_k a^{(k)2}$, which is the linear invariant of the crack density tensor α.

The effective moduli can be obtained from the elastic potential of the material with holes, by taking the derivative $\varepsilon_{ij} = \partial f/\partial \sigma_{ij}$. In the case of circular holes, the material is isotropic since $\beta_{ik} = p\delta_{ik}$. The effective moduli are

$$E = \frac{E_0}{1+3p}; \quad \nu = \frac{\nu_0 + p}{1+3p}. \tag{8}$$

When cracks are randomly oriented, isotropic effective properties can be expected, which means that β is proportional to a unit tensor. Since $tr\beta = 2p + q$, we have $\beta_{ik} = (p + q/2)\delta_{ik}$, and for cracks, $p = 0$ and $q = \pi\rho$, where ρ is the scalar crack density parameter. The moduli obtained are

$$E = \frac{E_0}{1+\pi\rho}; \quad \nu = \frac{\nu_0}{1+\pi\rho}. \tag{9}$$

2.3. *Interacting elliptical holes*

The determination of effective elastic properties of materials containing holes becomes significantly more complex when interactions between holes cannot be neglected. Several approximate schemes have been proposed, which can be divided in two main groups: the methods of effective matrix and the methods of effective field.

The methods of effective matrix place each hole in a medium with effective elastic properties. The effect of each individual hole on the moduli is obtained by considering it as an isolated one in the matrix with reduced stiffness. As a consequence, these methods predict that the impact of interactions always results in softening of the material as compared to the non-interacting approximation. Two most frequently used schemes of this type, the self-consistent scheme and the differential scheme, are discussed in the text to follow.

The method of effective field, which was first applied to composite materials, consists of placing a representative hole into the undamaged matrix and subjecting it to an effective stress field. It allows one to consider very general problems, because the effective stress field can be inhomogeneous, thus incorporating some information on mutual positions of holes. In this sense, hole interactions are not predicted to always produce a softening impact, which makes this method somewhat more powerful than the effective matrix methods. The method of Mori-Tanaka (1973), also used in the mechanics of composite materials, was applied to cracked materials by Benveniste (1986). It is in fact a simplified version of the method of effective field, in which the effective field is taken to be homogeneous and equal to its volume average, which considerably simplifies the calculations.

Self-consistent scheme. This scheme, formulated by Hill (1965) for the general case of composite materials, was applied by Budiansky and O'Connell (1976) to materials with randomly oriented cracks, and was extended by Hoenig (1979) to parallel cracks. In this method, the effect of interactions between holes is included by placing each hole separately in a medium with the effective elastic properties of the body of reduced stiffness. For a two dimensional isotropic material containing circular holes in random positions, the effective moduli obtained are

$$E = E_0(1-3p); \quad v = v_0\left[1-\left(3-\frac{1}{v_0}\right)p\right]. \tag{10}$$

In the case of randomly located cracks, the moduli can be expressed in terms of the scalar crack density parameter as

$$E = E_0(1-\pi\rho); \quad v = v_0(1-\pi\rho). \tag{11}$$

These formulae predict a much softer response than other approximate schemes. This effect is not observed in its modified version, called the generalized self-consistent scheme (Jun and Jasiuk, 1993).

Differential scheme. In this scheme, as in the self-consistent, the analysis is reduced to one isolated hole in the effective matrix. The difference between the two schemes, however, is that this type of analysis is done incrementally. The density parameters are increased in small steps, and the effective elastic moduli are recalculated at each step (Zimmerman, 1984; Kachanov, 1993). Another derivation of the differential scheme has been provided by Norris (1985), and it accounts for the possibility of hole overlap. This adjustment has not been considered in obtaining the subsequent expressions.

In the case of a two dimensional isotropic material containing randomly located circular holes, the results are

$$E = E_0\,e^{-3p}; \quad v = v_0\,e^{-(3-1/v_0)p}. \tag{12}$$

In the case of randomly located cracks the equations reduce to

$$E = E_0\,e^{-\pi\rho}; \quad v = v_0 e^{-\pi\rho}. \tag{13}$$

As is characteristic of all methods of effective matrix, these schemes always predict a softening effect due to interactions, and information on mutual positions of cracks is very difficult to incorporate. However, the softening effect predicted by the differential scheme is generally weaker than the effect predicted by the self-consistent scheme.

Method of effective field (method of Mori-Tanaka). As already mentioned, the method of effective field places a representative hole into an effective stress field, which generally does not coincide with the remotely applied field; the difference between these fields accounts for the effect of interactions.

In the case of randomly located elliptical holes, we can consider each hole as an isolated one subjected to an effective stress field that equals the average stress in the solid phase. The problem of a solid containing N holes is then solved as a superposition of N problems, each one containing one hole, where the effect of interactions between holes reflects the change of the average stress in the solid phase.

The effective moduli obtained for the case of randomly located elliptical holes are

$$E = \frac{E_0}{1+(3p+q)(1-p)^{-1}}; \quad \nu = \frac{\nu_0+p}{1+(3p+q)(1-p)^{-1}}. \tag{14}$$

It can be noticed that the factor $(1-p)^{-1}$, that accounts for the interactions effect on the effective moduli, causes softening of the material as compared to the non-interacting approximation. In the case of randomly located cracks, the results from the non-interacting approximation are recovered.

3. EXPERIMENTS

3.1. *Description*

Despite the number of different theoretical approaches proposed for determining effective elastic properties of solids, not much experimental work in this area could be found in the literature. Litewka (1985, 1986) performed tests with aluminum specimens containing sets of rectangular openings arranged in square patterns. Oda *et al.* (1984) used gypsum plaster samples with random cracks for unconfined compression and ultrasonic velocity tests and compared them with theoretical results formulated in terms of a fabric tensor (similar to the crack density tensor α introduced earlier by Kachanov, 1980). Vavakin and Salganik (1975) conducted uniaxial tension tests with rubber sheets containing randomly located rectilinear cracks and circular holes. However, as pointed out by Kanaun (1980), their arrangements of holes and cracks were not actually random, which may have affected the results.

The main purpose of this work is to determine the Young's modulus of an isotropic elastic material containing inhomogeneities—cracks or circular holes—located in random positions. This was accomplished by subjecting an artificially cracked, inhomogeneous elastic material to a uniaxial tensile stress under plane stress conditions. The specimens used were four identical rectangular aluminum (alloy 2024) plates with a length to width ratio of 3 : 1. The plates were 1.6 mm (0.063 in) thick, with a plane area of 685.8 mm × 228.6 mm (27.00 in × 9.00 in).

During the tests, the plates were pulled in tension along their length through two small holes approximately 6 mm in diameter located at a distance of 12 mm from the top and bottom edges. The holes were located with an accuracy of 0.03 mm to eliminate eccentric loading. The tensile load was applied through two steel pins of approximately the same diameter that were placed through these holes.

In order to obtain a one-dimensional homogeneous state of stress in the absence of inhomogeneities, only the central third part of the plates, corresponding to an area of approximately 228.6 mm × 228.6 mm, was monitored during the tests. It was shown that the distribution of stresses along that region was approximately homogeneous by measuring the strain along the section. The difference between the strains measured at the center of the section and close to the edges was less than 5%.

Two LVDTs, one at each side of the plate, were used to measure the vertical displacements corresponding to the central part of the plates. The value of strain used for calculation of the effective Young's modulus was obtained from averaging the responses of the two LVDTs and dividing by an appropriate gage length. Figure 2 shows the geometry of the plates without the inhomogeneities. The dashed lines delimit its central third part where the displacements were measured. The devices used to hold the LVDTs were made of steel and were designed to avoid movements at the plate connection during the tests.

Two different types of inhomogeneities were considered: slots (simulating cracks) and circular holes. These features were cut through the thickness of the plates to characterize the plane stress condition. Two of the plates contained slots and the other two contained circular holes. Slots were used instead of cracks due to the difficulty in creating real cracks in the plates. The minimum width of the slots that could be easily machined was 1.2 mm, for the plate thickness of 1.6 mm, and the lengths were calculated such that a minimum aspect ratio of 1 : 10 was obtained, in order to have crack-like features. For this aspect

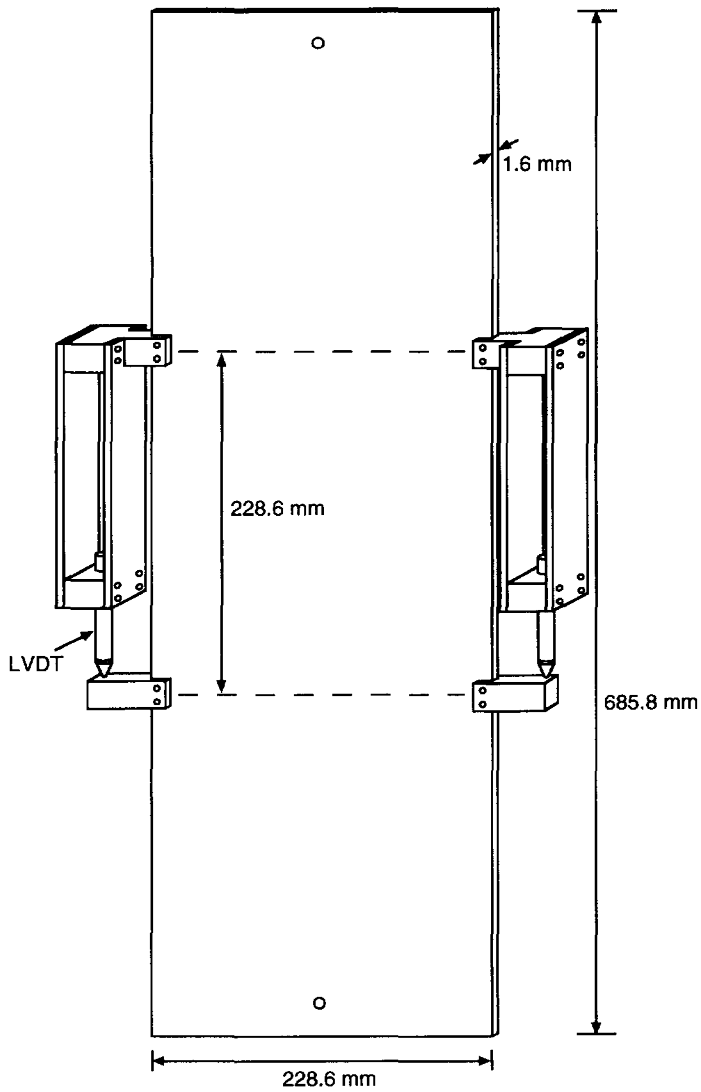

Fig. 2. Geometry of specimens.

ratio, the difference in the compressibility of the slots as compared to the case of cracks is only 1%, as shown in Fig. 1.

A random number generator was used to create the positions of the centers of the slots and holes over the central third of the plates. The x and y coordinates of each center were progressively generated such that, if a slot or hole would intersect another one already existing, it would be discarded and a new one would be generated. This procedure was used until the desired number of features were obtained without intersecting each other. In the case of slots, the orientations were not random, but prescribed to uniformly vary from 0 to 180 degrees. The purpose of this was to assure that an approximate isotropic distribution would be obtained, since the dimensions of the plates presented a limitation on the number of slots that could be used. To ensure a representative behavior of the nonhomogeneous element (the region over which displacements were measured) slots or holes were also located outside the central part by reflection of the positions already existing. Figures 3 and 4 show the positions of the slots and holes in the plates.

Several tests were performed to determine the variation of the Young's modulus with crack density (for plates containing slots), and with porosity (for plates containing holes). The two plates containing slots were tested for the same values of crack density, but with different crack distributions. To investigate the minimum number of slots needed to characterize a random array, one of the plates had six slots, while the other one had 20

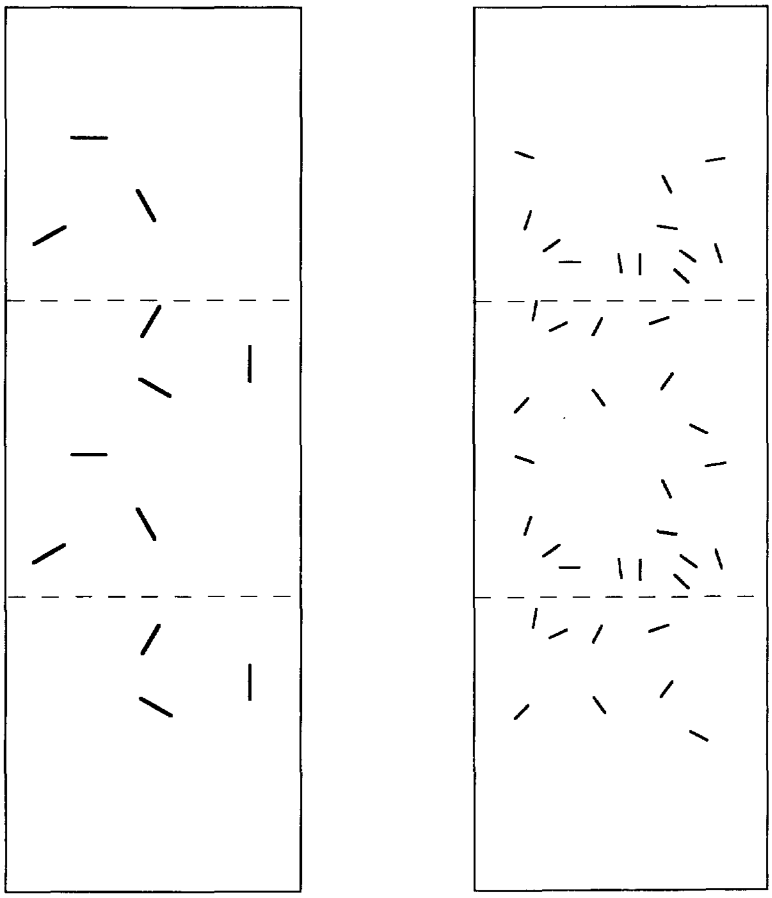

Fig. 3. Layout of the plates for $\rho = 0.02$.

slots. For each value of crack density all the slots in each plate had the same length. The plates were tested for values of crack density between 0.02 and 0.15. The same procedure was followed for the plates containing holes; one of the plates had six holes and the other had 20. Each plate was tested for values of porosity between 0.02 and 0.13. Each test was repeated three times, and the data obtained were then compared to the theories presented in the previous sections.

3.2. *Calibration tests*

Calibration tests were performed before any slots or holes were cut into the plates and had two main purposes: determination of the Young's modulus of the intact material and determination of a gage length for the effective properties experiments. This last test corresponds to finding the exact distance over which displacements are being measured by the LVDTS. This can be justified by the fact that the need for two different fixing points around the top and bottom dashed lines in Fig. 2 might cause that length to differ from its first estimation (one third of the total length of the plate).

These tests were performed using one of the plates designed for the actual tests, but without any inhomogeneities. Tensile loads up to 4.5 kN were applied to the plate and the corresponding strains were obtained from two strain gages located on opposite sides of the plate at the centre of its plane area. The plate was tested five times, and the maximum difference between the values obtained for Young's modulus was less than 0.5%. The Young's modulus of the intact material was taken as 71.8 GPa, which is the average of those values. This value also is in very good agreement with the Young's modulus obtained from uniaxial compression tests for this same type of material (72.0 GPa).

The gage lengths were obtained from those same tests by dividing the average of the LVDTs' readings by the average of the strain gages readings (to eliminate the effect of

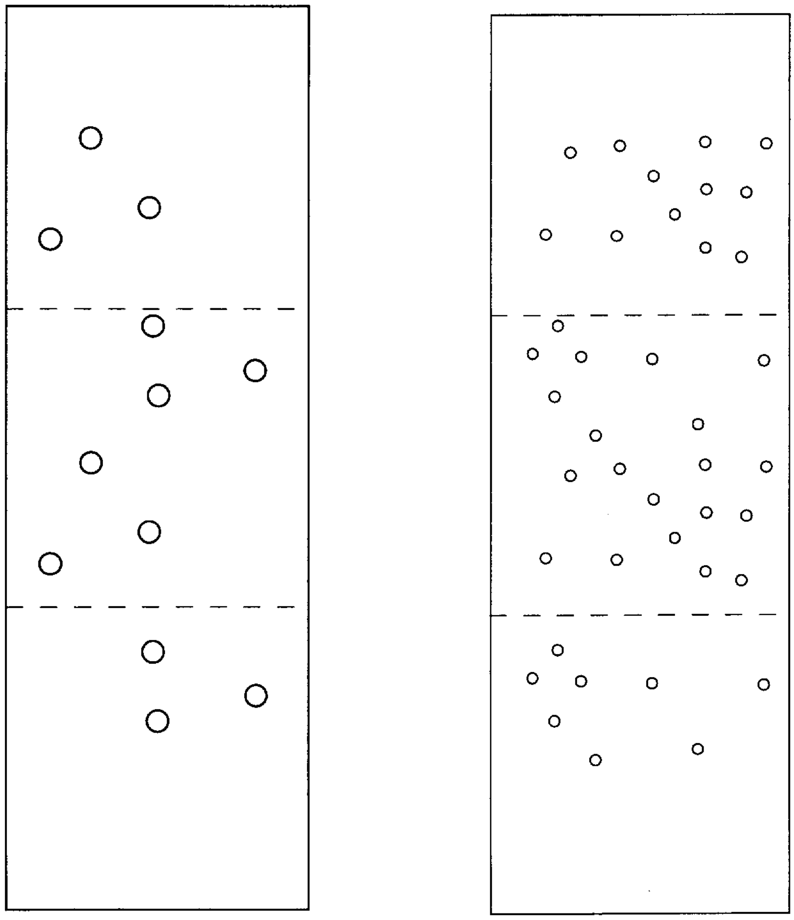

Fig. 4. Layout of the plates for $p = 0.02$.

bending). Comparison of the values obtained from different tests showed a maximum difference of 0.5%. The gage length was taken as the average between them (246.6 mm), which is actually greater than the originally estimated value of one third of the total length (228.6 mm). Geometrically, the value obtained for the gage length corresponds approximately to the outside distance between the outer screws that fix the LVDTs' holding devices (243 mm). The small difference between these values, on the order of 2%, can be attributed to the presence of friction in the contact between the LVDTs' holding devices and the plate itself.

The sensitivity of the load cell used was 1.25 kN/V, and the sensitivity of the LVDTs was approximately 0.1 mm/V. Because of the extremely small displacements that were expected during the tests (on the order of 0.01 mm) the sensitivity of the LVDTs turned out to be an important factor in the evaluation of the results. The behavior of the LVDTs displayed a variability not greater than 0.1%.

3.3. *Results*

The calibration tests provided values for the gage length of the displacement measurements and the Young's modulus of the intact material. Note that the gage length was needed in the calculations of the crack density (for the plates containing slots) and the porosity (for the plates containing holes).

In the tests performed, the readings obtained from the two LVDTs were averaged, and these values were divided by the gage length to obtain the average strain of the element. The stresses were obtained by dividing the applied loads by the cross-sectional area of the plates (365.8 mm^2). Dividing the axial stresses by the average axial strains yields the effective

Young's modulus of the inhomogeneous material. The results for both cases of plates containing slots and holes are presented in terms of a ratio between the effective Young's modulus (E) and the Young's modulus of the intact material (E_0) as a function of the appropriate density parameter (crack density or porosity).

An error analysis was made based on the results obtained from the calibration tests for the Young's modulus of the material and the gage length. For this analysis, it was assumed that there was some variability associated with the readings from the strain gages, as well as with the readings from the LVDTs. No error associated with the applied loads was included. As a result, the error associated with the determination of the ratio between the effective Young's modulus and the Young's modulus of the intact material was not greater than 0.8%. Because this number is very small, no error bars could be shown.

Each plate was tested for tensile loads up to 4.5 kN. At loads of 5 kN or more, some permanent deformation around the holes through which the loads were applied could be observed. All plates showed elastic behavior during the tests for the loads applied.

Figure 5 shows the results of the tests performed with the plates containing slots. In order to judge the influence of the slots located outside the central area of the plates, the first test was performed without them. The slots outside were then cut and the test was repeated. As observed, the difference in the results was considerable. The following tests were performed with the outside slots, thus ensuring that the behavior of a representative element was being measured.

As expected, the effective Young's modulus of both plates decreases as the crack density increases. It can be observed, however, that for the same values of crack density the plate containing six slots has always a higher effective Young's modulus than the plate containing 20 slots. The six-slots configuration was checked with a numerical calculation (Shah *et al.*, 1994) and the results showed predominance of stress shielding for that particular array at lower densities. This explains the experimental results, and suggests that 6 slots may not be a sufficient number to characterize a random distribution.

The topmost line in the graph represents the analytical results obtained from the approximation of non-interacting cracks, eqn (9), which coincides with the predictions of Mori-Tanaka's scheme. The experimental results are in very good agreement with the non-interacting theory even at higher values of crack density, where interactions probably occur. The differential scheme predictions are also close to the experimental results for lower values of crack density. However, at high densities, the non-interacting approximation provides better predictions. The experimental results are not in good agreement with the predictions of the self-consistent scheme, which tends to overestimate the effective compliance.

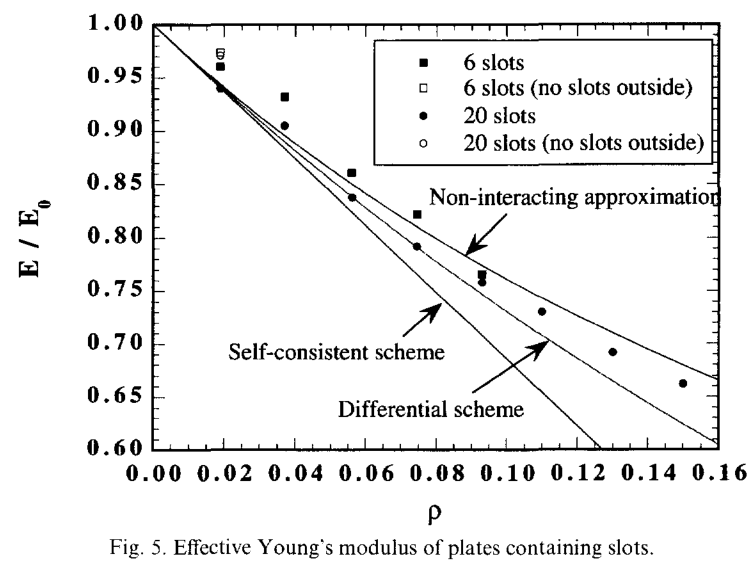

Fig. 5. Effective Young's modulus of plates containing slots.

The results for the plates containing circular holes are shown in Fig. 6. The first two tests (for values of porosity of 0.02 and 0.05) were performed without any holes outside the central area of the plates. The holes were then added and the second test was repeated (porosity of 0.05). Again, a considerable difference in the results can be observed, and subsequent tests were performed with holes outside the central region to ensure that a representative behavior was measured.

The lines in the graph show the analytical results presented in the previous section for the cases of interacting and non-interacting circular holes. It can be observed that, as porosity increases, and interactions become more likely, the results for both plates follow the interacting approximation. The results also suggest that at $p = 10\%$, interactions already occur in the plate containing 6 holes, but not in the one with 20 holes. At all other values of porosity, the results for both plates are very close and in very good agreement with the theoretical predictions. This can be explained by the fact that, for the case of circular holes, the only parameter that accounts for the presence of holes in both interacting and non-interacting cases is porosity, which is insensitive to the type of distribution (random or not). Therefore, the number of holes in each plate, for each approximation, should not influence the results if both plates have the same value of porosity.

As in the case of cracks, the self-consistent scheme seems to overestimate the effective compliance of the plates. The experimental results obtained were not able to distinguish between the predictions of Mori-Tanaka's and the differential scheme, for values of porosity up to 13%. This same behavior has been observed by Zimmerman (1991) when analyzing experimental data on 3-D materials containing spherical pores.

4. CONCLUDING REMARKS

The effective Young's modulus of plates containing slots or circular holes located in random positions was measured. The ratio between the effective Young's modulus of the plates and of the intact material was shown as a function of the proper density parameter for each case: crack density, for slots, or porosity, for circular holes. The results obtained were compared with some of the existing analytical expressions.

In the case of plates containing slots, the approximation of non-interacting cracks is a reasonable one, even at relatively high values of crack density, where interactions are expected to occur. For the plates containing circular holes, it was observed that the effective Young's modulus approaches the predictions of an interacting approximation as porosity increases. In both cases considered (slots and circular holes), the predictions of the self-consistent scheme did not show good agreement with the experimental results, over-estimating the effective compliance. In the case of circular holes, for the values of porosity

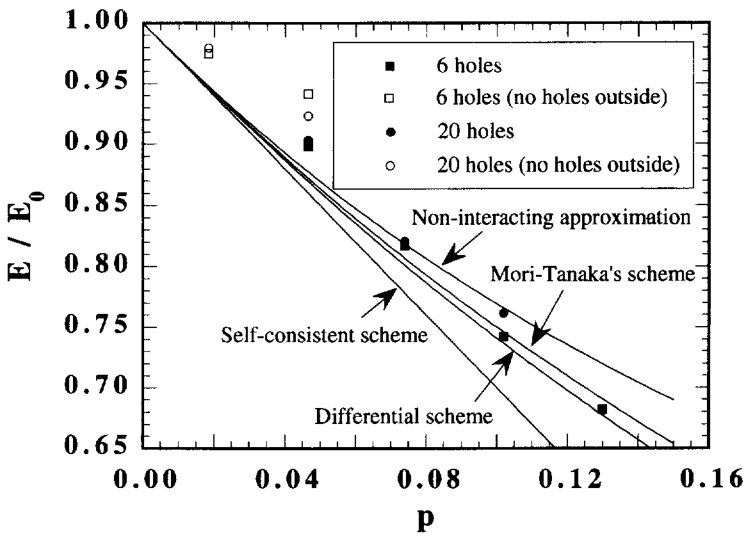

Fig. 6. Effective Young's modulus of plates containing circular holes.

used, the results were not able to distinguish between the predictions of the Mori-Tanaka scheme and the differential scheme.

The results obtained should be considered only within the context of the experiments (the number of samples examined and the densities of defects tested). However, the good agreement obtained between the theoretical analysis and the experimental results suggests that the tests are appropriate for measuring effective Young's modulus of elastic materials containing randomly located inhomogeneities under plane stress conditions. In the case of plates containing slots, the results also suggest that a minimum number is needed, certainly greater than six and probably at least 20, to guarantee randomness of locations.

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